Marvin Paul Pastel

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by

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RESEARCH PAPER NO. 23

UNITED STATES NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

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Introduction

The characteristics of a linear component are known to be completely determined by its steady state frequency response which relates the output of the component to its input. Hence, the analysis and synthesis of a complex system are simplified when the frequency response of each unit making up the system is known. For such a system, the overall amplitude ratio of output to input for any frequency is obtained by multiplication of the amplitude ratios for each individual component. As a result of this property, a lsrge body of design procedure for control systems has been based on linear frequency response methods.

While the experimental measurement of frequency response for electronic amplifiers, electric motors, and other electrical equipment is generally not difficult, the application of a sinusoidal input with constant amplitude and frequency to hydraulic or pneumatic components is often impractical. When the nature of the physical component prohibits measurement of the frequency response, a transient response test (to a suitable step or impulse input function) may be permissible. If a method were available by which the frequency response could be derived from this transient response, frequency response design techniques could be applied for the system incorporating this component.

The mathematical foundation for such a method would be the Fourier transform or the Laplace transform, with the complex number restricted to values of $j\omega$. When the time response is not in analytical form,

numerical methods of applying the Fourier or Laplace transforms must be used. The rest of this paper will describe such a method. The method requires that the input function be known analytically and that both the input and the component output functions tend to a constant rate of change after a finite time.

Deviation of Method

Figure 1 shows an output quantity vs. time response curve in which the time axis has been divided into equal intervals and straight lines drawn between the corresponsing points on the output curve. This approximation of the curve by straight line segments can be as exact as desired by reducing the size of the intervals. Further, the approximation can be decomposed into a series of straight line curves. This is shown in Figure 2. In this figure, the slopes m_o, m_1, etc. are of such a value that the approximate analytic expression for the output becomes the series.

$$x(t) = m_o t + m_1[t-\sigma] + m_2[t-2\sigma] + \cdots + m_{n-1}[t-(n-2)\sigma] + m_n[t-(n-1)\sigma] + \cdots \quad (1)$$

The Laplace transform for each component of the series (1) may be taken and the sum representing the transform of the approximate output transient may be written.

$$X(s) = \frac{m_0}{s^2} + \frac{m_1}{s^2} e^{-\sigma s} + \frac{m_2}{s^2} e^{-2\sigma s} + \cdots + \frac{m_{n-1}}{s^2} e^{-(n-2)\sigma s} + \frac{m_n}{s^2} e^{-(n-1)\sigma s} + \cdots \tag{2}$$

In a typical application of (2), it would be more convenient to express the slopes m in terms of the output quantity and time. The first two slopes in terms of x and t are:

$$m_0 = \frac{X_1}{\sigma}$$

$$m_1 = \frac{X_2 - X_1}{\sigma} - \frac{X_1}{\sigma} \tag{3}$$

The value for m_2 is the slope of the third segment less the slope of the second segment.

$$m_2 = \frac{X_3 - X_2}{\sigma} - \frac{X_2 - X_1}{\sigma}$$

$$= \frac{X_3 - 2X_2 + X_1}{\sigma} \tag{4}$$

Then the general term becomes

$$m_K = \frac{X_K - 2X_{K-1} + X_{K-2}}{\sigma} \tag{5}$$

Equation (2) in terms of the ordinate of the output curve for simplicity is written as a summation.

$$X(s) = \frac{1}{\sigma s^2}\left\{X_1 + \sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})e^{-(K-1)s}\right\} \quad (6)$$

The expression for the transfer function can be obtained by dividing (6) by Laplace transform of the input y.

$$\frac{X}{Y}(s) = G(s) = \frac{1}{Y(s)\sigma s^2}\left\{X_1 + \sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})e^{-(K-1)s}\right\} \quad (7)$$

The frequency response is found by substituting $j\omega$ for s in (7).

$$G(s) = \frac{1}{Y(j\omega)\sigma\omega^2}\left\{X_1 + \sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})e^{-(K-1)j\omega}\right\} \quad (8)$$

Euler's relation

$$e^{-jb} = \cos b - j\sin b$$

applied to equation (8) gives

$$G(j\omega) = \frac{1}{Y(j\omega)\sigma\omega^2}\left\{X_1 + \sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})\cos(K-1)\sigma\omega - j\sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})\sin(K-1)\sigma\omega\right\} \quad (9)$$

Conveniently generated input time functions are the impulse, step displacement and step velocity (ramp) with corresponding Laplace transform 1, 1/s, $1/s^2$

Substituting these in (9) gives the following useful equations.

Impulse:

$$G(j\omega) = -\frac{1}{\sigma\omega^2}\left\{X_1 + \sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})\cos(K-1)\sigma\omega \right. \\ \left. -j\sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})\sin(K-1)\sigma\omega \right\} \tag{10}$$

Step displacement:

$$G(j\omega) = -\frac{1}{\sigma\omega}\left\{\sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})\sin(K-1)\sigma\omega + \right. \\ \left. j\left[X_1 + \sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})\cos(K-1)\sigma\omega\right\}\right. \tag{11}$$

Step velocity:

$$G(j\omega) = -\frac{1}{\sigma}\left\{X_1 + \sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})\cos(K-1)\sigma\omega \right. \\ \left. -j\sum_{K=2}^{\infty}(X_K - 2X_{K-1} + X_{K-2})\sin(K-1)\sigma\omega \right\} \tag{12}$$

Dead time, the time difference between the initiation of an input to the system and the beginning of the output response, is handled automatically by the above method.

Referring to equation 8, the first term will be zero if the time increment is less than the dead time and then all terms will contain the exponential factors representing linear added phase shift corresponding to a time delay.

Example

As an illustration of this method one example is used based on analytically determined system response curves for which the actual frequency response is determined for comparison. A second-order system transfer function

$$G(s) = \frac{\omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2} \tag{13}$$

is shown in Figure 4 for the critically damped case with ω_o set equal to unity. This curve can be obtained directly by replacing s by $j\omega$ in equation 13.

Setting $\zeta = 1$

$$G(s) = \frac{1}{-\omega^2 + 2j\omega + 1}$$

$$= \frac{1-\omega^2}{(1+\omega^2)^2} - j\frac{2\omega}{(1-\omega^2)^2} \tag{14}$$

and the time response of the system with

to an impulse is given by

$$\chi = te^{-t} \tag{15}$$

Equation 15 is plotted in Figure 3. Choosing σ as one half second, the required values of x substituted into equation 10 gives

$$G(j\omega) = -\frac{2}{\omega^2}\Big\{0.303 - 0.238\cos(0.5\omega) - 0.098\cos\omega -$$
$$0.032\cos(1.5\omega) + 0.010\cos(2.5\omega) + 0.010\cos(3\omega)$$
$$+ 0.010\cos(3.5\omega) + 0.013\cos(4\omega) + 0.010\cos(5\omega)$$
$$+0.012\cos(6.5\omega) - j\big[0.238\sin(0.5\omega) - 0.098\cos\omega$$
$$- 0.032\sin(1.5\omega) + 0.01\sin(2.5\omega) + 0.01\sin 3\omega$$
$$+ 0.01\sin(3.5\omega) + 0.013\sin(4\omega) + 0.015\sin(5\omega)$$
$$+ 0.012\sin(6.5\omega)\big]\Big\}$$

The equation was solved for frequencies 0.1, 0.3, 0.7, 1 and 2 radians per second and the values obtained are indicated on Figure 4.

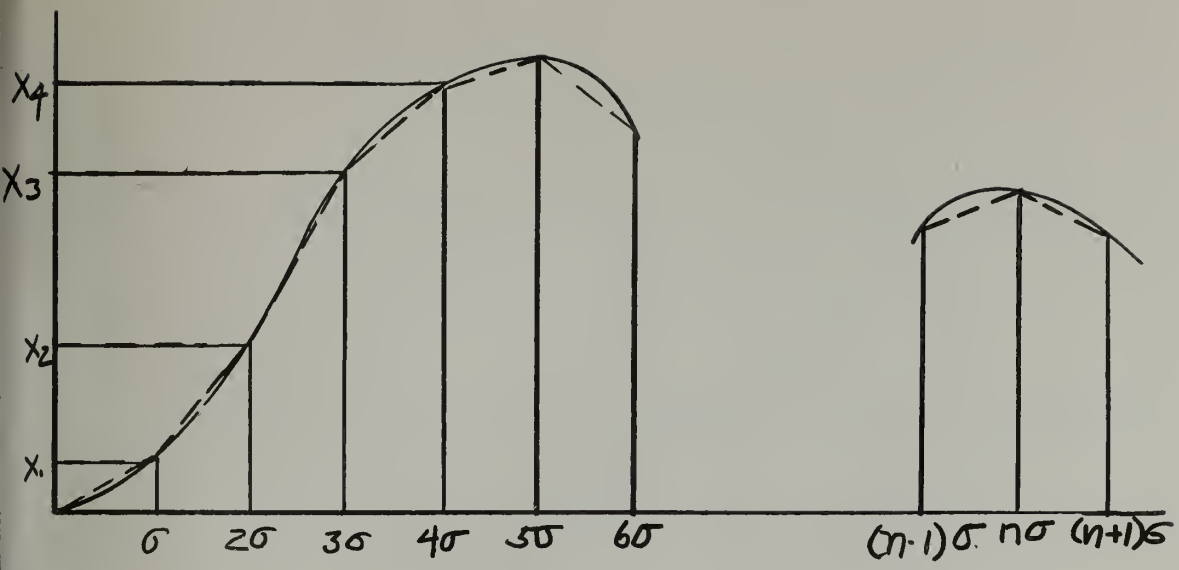

Figure 1. Transient response and straight line approximation

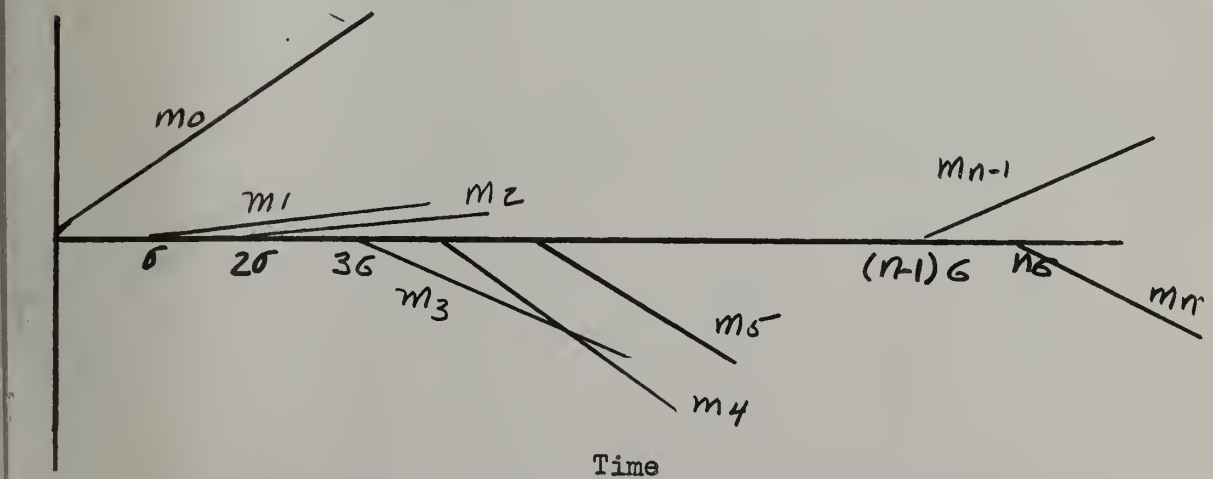

Figure 2. Components of the transient

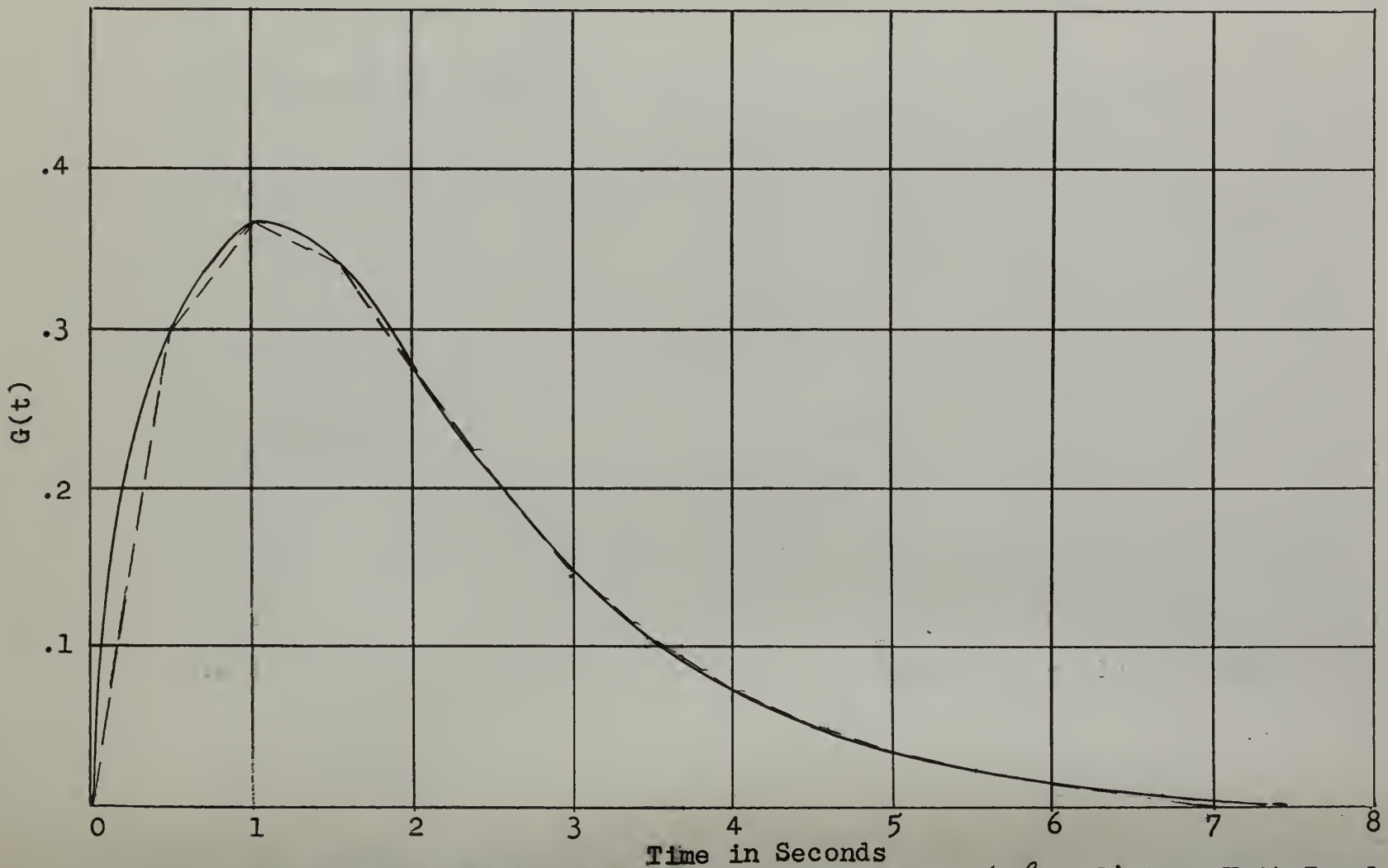

Figure 3 Transient Response at Second Order System (ζ = 1) to a Unit Impulse

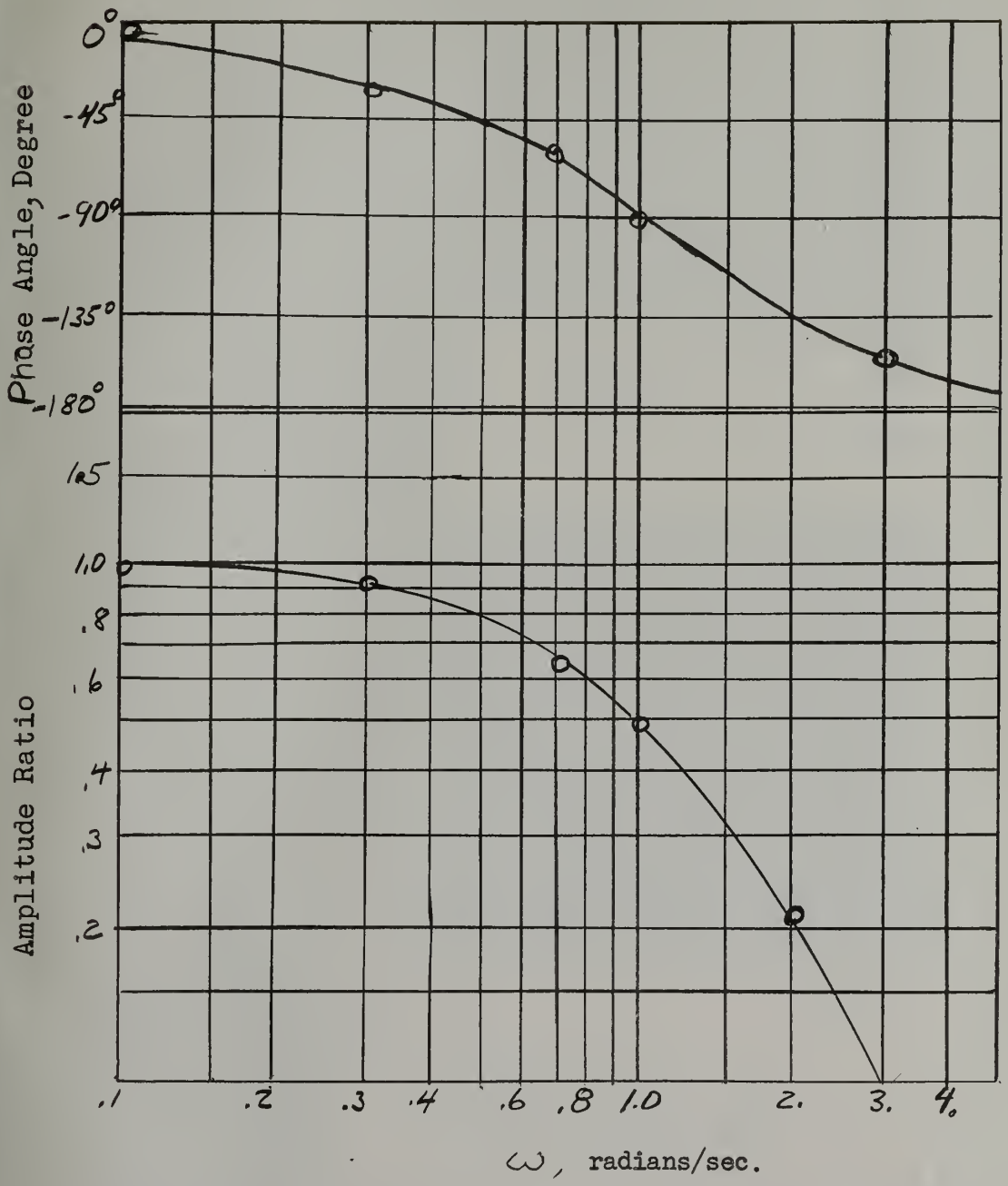

Figure 4 Frequency response for second order system ($\zeta = 1$) compared with computed points

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